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APPROXIMATE EXPRESSIONS FOR THE LUMINOSITY AND R.M.S. DIAMOND LENGTH FOR EQUAL COLLIDING BEAMS

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ABSTRACT

For equal colliding beams, approximate expressions are derived for the luminosity and r.m.s. diamond length. The region of validity for these expressions is examined by comparison with numerical results from exact integrals.

I. Introduction

When considering the construction of any accelerator, the luminosity and r.m.s. diamond length are of central importance. In this paper approximate expressions for these quantities will be derived. These new expressions will be quite adequate for most calculations, there being regions where their deviation from the exact theory is negligible. It turns out, in fact, that over a wide range of values of the systems relevant parameter σ_ℓ/β_H^* there is excellent agreement.

In this paper, the above discussion will be amplified. In Section II an approximate expression for the luminosity will be derived which is valid for the most general case of a non-zero crossing angle. In Section III an approximation to the r.m.s. diamond length is developed. To do the above things we essentially approximate the luminosity, or "overlap," integrand. Section IV contains the comparisons between the new approximate formulas and the exact theory as well as the conclusions of this work.

II. The Luminosity

For the case where both colliding beams are bunched and the assumption that the lattice has zero dispersion at the crossing point the most general expression for the luminosity is given by[1),2)] $L = N_1 N_2 \, f_{encounter} \, F$ where

$$F = \frac{2}{(2\pi)^{3/2}(\sigma_{\ell_1}{}^2+\sigma_{\ell_2}{}^2)^{1/2}} \int_{-\infty}^{+\infty} \frac{dS \; e^{-2S^2\left[\frac{1}{\sigma_{\ell_1}{}^2+\sigma_{\ell_2}{}^2} + \frac{\alpha^2/4}{\sigma_{x_1}{}^2+\sigma_{x_2}{}^2}\right]}}{(\sigma_{x_1}{}^2+\sigma_{x_2}{}^2)^{1/2}(\sigma_{z_1}{}^2+\sigma_{z_2}{}^2)^{1/2}} \; . \qquad (1)$$

$f_{encounter}$ is defined to be the frequency of encounter and is equal to $f_o B$ where f_o = frequency of revolution = $C/2\pi R$ where C = velocity of particle and R = radius of ideal orbit, B = # of bunches per ring and N_1, N_2 are defined to be the number of particles per bunch in beams 1 and 2 respectively.

We are interested in deriving an approximate expression for equation (1) that will be both accurate and simple to apply. Firstly we should state our expectations for our new expression. We expect the ratio of the exact formula to the approximate one to approach 1 as σ_ℓ/β_H^* approaches 0, and to approach 0 as σ_ℓ/β_H^* approaches infinity. In other words the approximation should be best in the region that makes the dominant contribution to the luminosity integral, which is the origin, and not so good in the regions making negligible contributions. We therefore will look at the denominator in equation (1) and approximate it as an exponential which can then be evaluated explicitly.

For equal beams equation (1) becomes

$$F = \frac{\sqrt{2}}{\sigma_\ell (2\pi)^{3/2}} \int_0^{\infty} \frac{dS \; e^{-S^2\left[\frac{1}{\sigma_\ell^2} + \frac{\alpha^2/4}{\sigma_H^2}\right]}}{\sigma_H \sigma_v} \; . \qquad (2)$$

Now since $\sigma_{H,v}^2 \equiv \sigma_{H,v}^{*2}(1+S^2/\beta_{H,v}^{*2})$, and dropping terms in the denominator of order S^4 we re-write equation (2) as

$$F \cong \frac{\sqrt{2}}{\sigma_\ell (2\pi)^{3/2}\sigma_H^* \sigma_v^*} \int_0^\infty dS \; \frac{e^{-S^2\left[\frac{1}{\sigma_\ell^2} + \frac{\alpha^2/4}{\sigma_H^{*2}(1+S^2/\beta_H^{*2})}\right]}}{\sqrt{1+S^2\left(\frac{1}{\beta_H^{*2}} + \frac{1}{\beta_v^{*2}}\right)}} \; . \tag{3}$$

making a change of variables $S \to S'\sigma_\ell$ we see that in the limit $\sigma_\ell << \beta_H^*$

$$F \cong \frac{\sqrt{2}}{(2\pi)^{3/2}\sigma_H^* \sigma_v^*} \int_0^\infty dS' \; \frac{e^{-S'^2\left(1+\frac{(\alpha^2/4)\sigma_\ell^2}{\sigma_H^{*2}}\right)}}{\sqrt{1+\rho^2 \; S'^2}} \tag{4}$$

where

$$\rho^2 = \left(\frac{\sigma_\ell}{\beta_H^*}\right)^2 + \left(\frac{\sigma_\ell}{\beta_v^*}\right)^2 \; . \tag{5}$$

The contribution to the integral in equation (4) comes predominantly from the region $0<S'<1$ for which we take the approximation

$$F^{approx} \equiv \frac{\sqrt{2}}{(2\pi)^{3/2}\sigma_H^* \sigma_v^*} \int_0^\infty dS' \; e^{-S'^2 \left(1 + \frac{1}{2}\rho^2 + \left[\frac{\alpha\sigma_\ell}{2\sigma_H^*}\right]^2\right)} . \tag{6}$$

It's now trivial to show that

$$F^{approx} = \frac{1}{2\pi\sigma_H^* \sigma_v^*} \frac{1}{\sqrt{1 + \frac{1}{2}\left(\frac{\sigma_\ell}{\beta_H^*}\right)^2 + \frac{1}{2}\left(\frac{\sigma_\ell}{\beta_v^*}\right)^2 + \left(\frac{\alpha\sigma_\ell}{2\sigma_H^*}\right)^2}} \; . \tag{7}$$

F^{approx} can be used with confidence whenever the conditions $\sigma_\ell << \beta_H^*, \beta_v^*$ are satisfied. The errors one makes by using the approximate luminosity expression will be discussed later.

III. The R.M.S. Diamond Length

Now we wish to develop an approximate expression for the r.m.s. diamond length. To do this we approximate the integrand in equation (2) by

$$\frac{e^{-S^2\left[\frac{1}{\sigma_\ell^2}+\frac{\alpha^2/4}{\sigma_H^2}\right]}}{\sigma_H\sigma_v} = e^{-S^2/2\sigma_D^2} . \qquad (8)$$

A naive first approximation to the diamond length σ_D is obtained by making a change of variables $S \to \sigma_\ell S'$ and recalling that for $\sigma_\ell << \beta_H^*$ $\sigma_H^2 \to \sigma_H^{*2}$ and ignoring the denominator in equation (8). If we do this we immediately obtain

$$\sigma_D \cong \frac{\sigma_\ell}{\sqrt{2}} \frac{1}{\sqrt{1+(\alpha\sigma_\ell/2\sigma_H^*)^2}} . \qquad (9)$$

A better approximation can be obtained by not ignoring the denominator in equation (8). Using similar methods to those of Section II we arrive at the following expression:

$$\sigma_D^{\text{approx}} = \frac{\sigma_\ell}{\sqrt{2}} \frac{1}{\sqrt{1+\frac{1}{2}(\sigma_\ell/\beta_H^*)^2+\frac{1}{2}(\sigma_\ell/\beta_v^*)^2+(\alpha\sigma_\ell/2\sigma_H^*)^2}} . \qquad (10)$$

To arrive at an exact expression for the r.m.s diamond length we equate

$$\int_{-\infty}^{+\infty} \frac{dS\ e^{-S^2\left[\frac{1}{\sigma_\ell^2}+\frac{\alpha^2/4}{\sigma_H^2}\right]}}{\sqrt{(1+S^2/\beta_H^{*2})(1+S^2/\beta_v^{*2})}} = \int_{-\infty}^{+\infty} dS\ e^{-S^2/2\sigma_D^2} = 2\sqrt{2\pi}\sigma_D \qquad (11)$$

so that

$$\sigma_D^{(\text{theo.})} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dS\ e^{-S^2\left[\frac{1}{\sigma_\ell^2}+\frac{\alpha^2/4}{\sigma_H^2}\right]}}{\sqrt{(1+S^2/\beta_H^{*2})(1+S^2/\beta_v^{*2})}} . \qquad (12)$$

Looking at the formalism's of Sections II and III we notice that equations (7) and (10) are good approximations to equations (2) and (12) whenever

$$\int_{-\infty}^{+\infty} \frac{dS\ e^{-S^2\left[\frac{1}{\sigma_\ell^2} + \frac{\alpha^2/4}{\sigma_H^2}\right]}}{\sqrt{(1+S^2/\beta_H^{*2})(1+S^2/\beta_v^{*2})}} \cong \frac{\sigma_\ell\sqrt{\pi}}{\sqrt{1+\frac{1}{2}(\sigma_\ell/\beta_H^*)^2+\frac{1}{2}(\sigma_\ell/\beta_v^*)^2+(\alpha\sigma_\ell/2\sigma_H^*)^2}} \quad . \tag{13}$$

IV. Remarks and Conclusion

Figures (1) thru (4) are plots of the exact theoretical expressions/approximate ones vs. the relevant parameter σ_ℓ/β_H^*. As can be seen from the figures our expectations for the behavior of the ratio have met admirably. AL $= \alpha\sigma_\ell/2\sigma_H^*$ and Beta $H^*/v^* = \beta_H^*/\beta_v^*$. The curves labelled $\beta_H^*/\beta_v^* = 1.0$ are obtained by setting the vertical and horizontal terms equal. Notice that for AL = 0.0 this curve indicates excellent agreement of the approximate and exact expressions for $\sigma_\ell/\beta_H^* \leq 1.4$. For AL = 1.0 there is excellent agreement also for $\sigma_\ell/\beta_H^* \leq 1.1$, whereas AL = 2.0 and 5.0 show that the values of σ_ℓ/β_H^* which indicate excellent agreement are diminishing. The general trend seems to be that as AL increases for $\beta_H^*/\beta_v^* = 1$ our approximations region of validity is falling off. The curves labelled $\beta_H^*/\beta_v^* = 0.1$ are obtained by letting the horizontal terms dominate. They follow the same trend as the first set of curves, the region of validity of the approximation being only slightly smaller. The third set of curves is obtained by letting the vertical term dominate and has $\beta_H^*/\beta_v^* = 10$. The approximate formulas for the luminosity and r.m.s. diamond length have only limited validity here, the two sets of approximate and exact expressions diverging rapidly here.

In Conclusion:

1) We've derived approximate expressions for the luminosity and r.m.s. diamond length which contain higher order corrections to the most naive guess for such expressions.

2) We've examined the regions of validity for the newly derived expressions, the relevant parameter used to do this being σ_ℓ/β_H^*. Three separate cases were examined.

a) $\beta_H^*/\beta_V^* = 1.0$

b) $\beta_H^*/\beta_V^* = 0.1$

c) $\beta_H^*/\beta_V^* = 10.0$.

We found that cases a) and b) exhibited excellent agreement for a wide range of values of σ_ℓ/β_H^*, but noted that as AL increases the region of validity falls off rapidly. Case c) was seen to diverge much more rapidly than cases a) and b), warning us to use extreme caution when using the formulas developed in this paper for this particular case.

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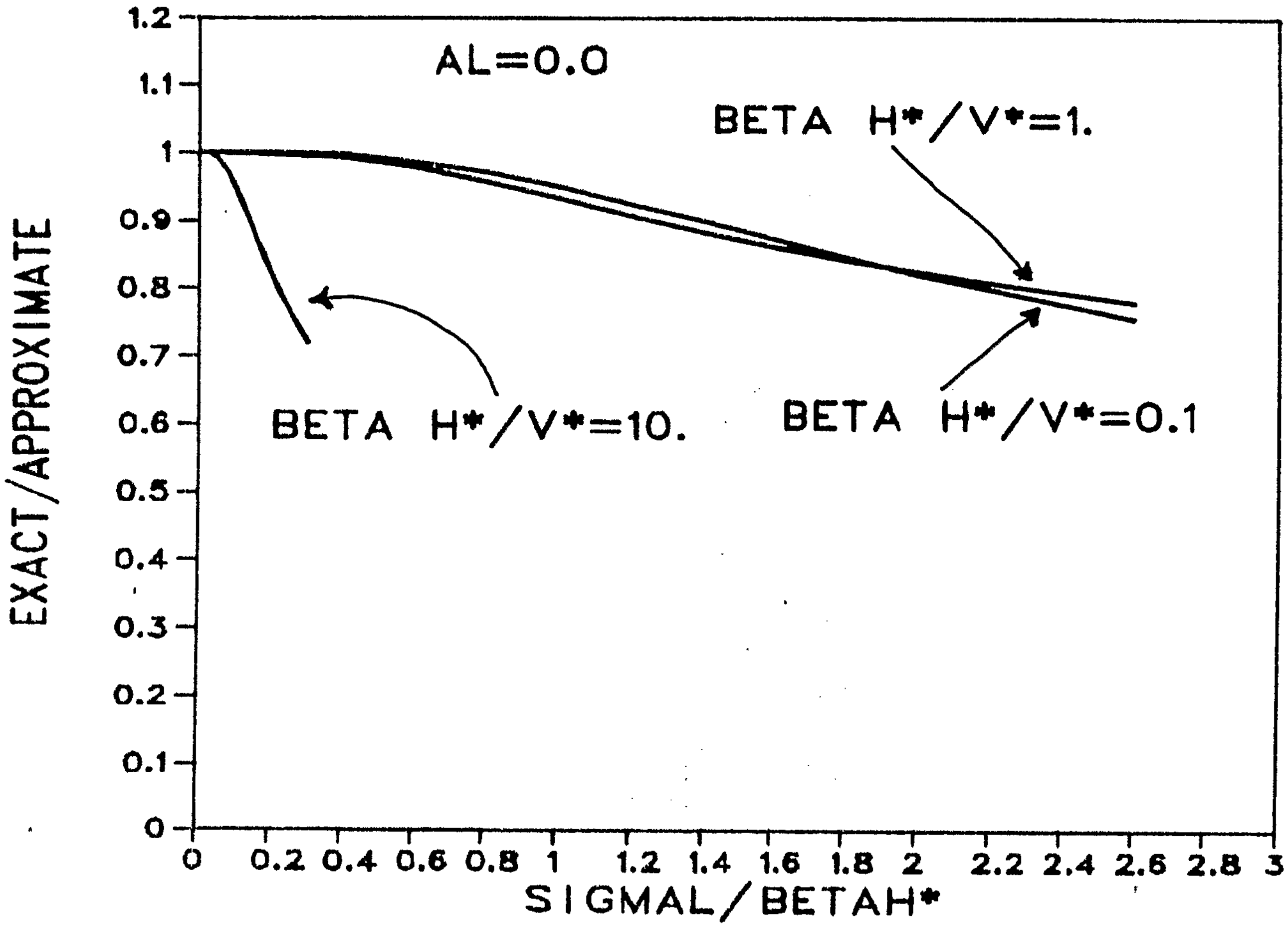
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SIGMAL/BETAH*
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1.1
1
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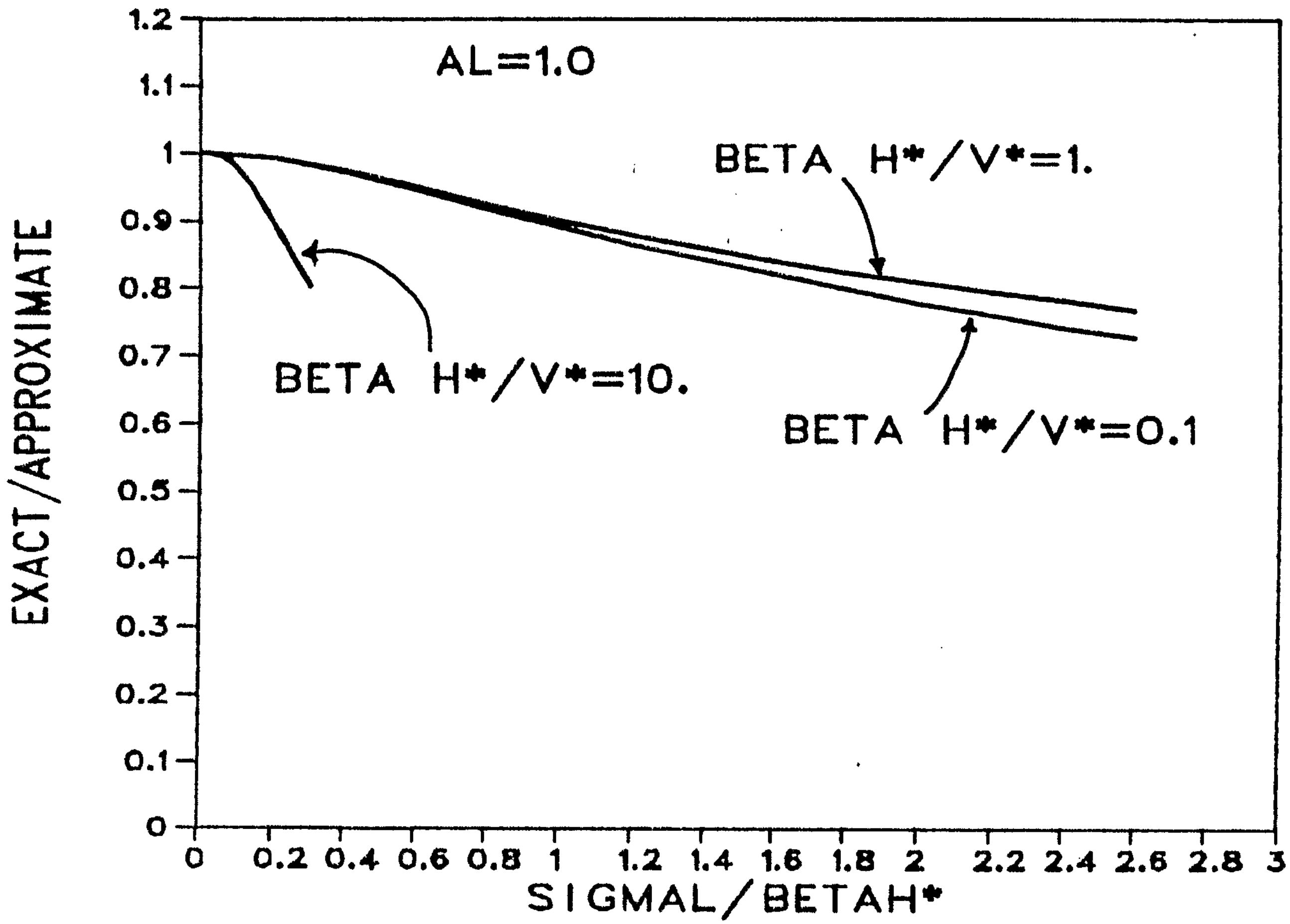
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2 2.2 2.4 2.6 2.8 3

Fig. 1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1
0.9
0.8
0.7
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0.5
0.4
0.3
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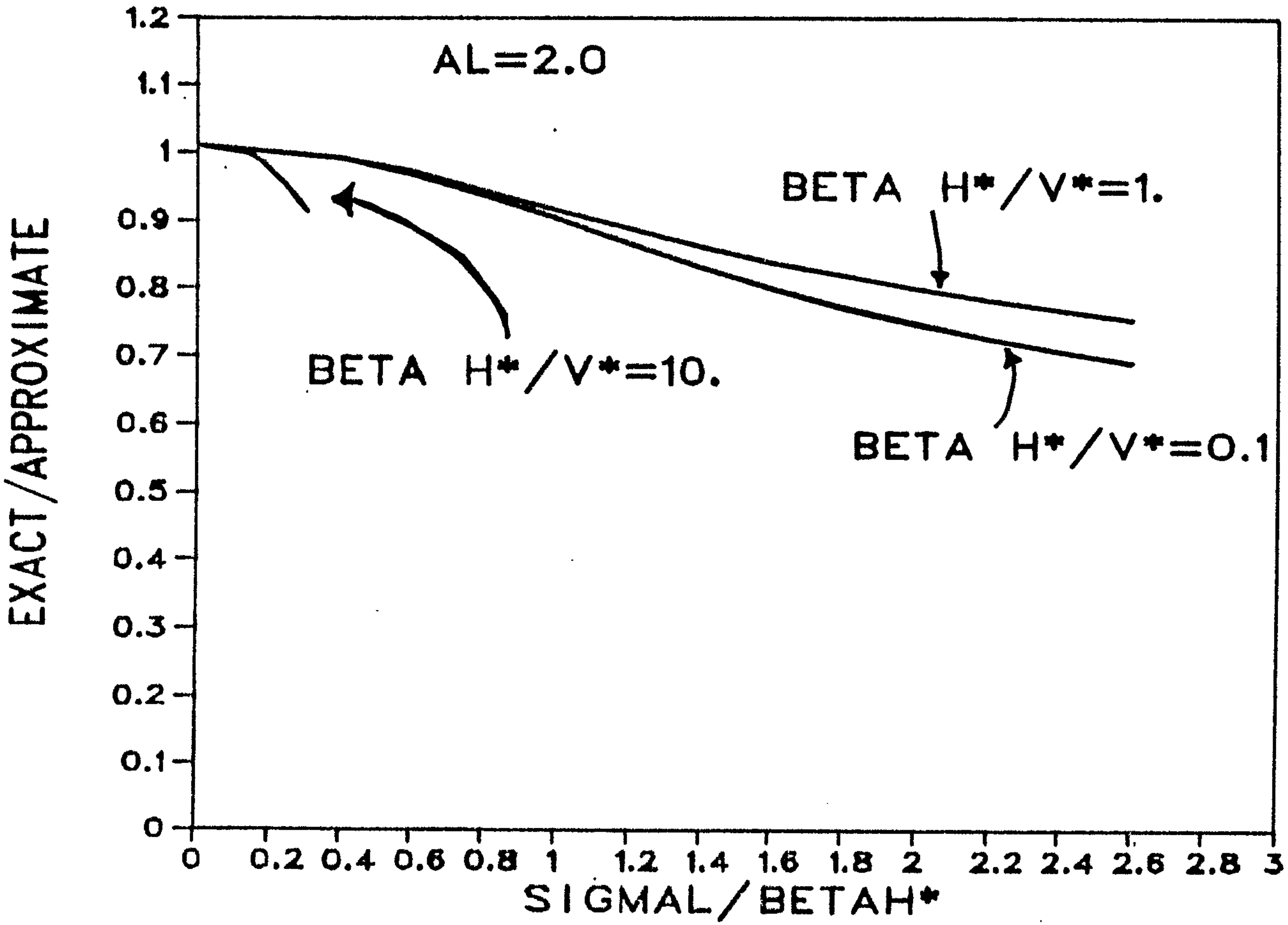
Fig. 2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0.8
0.7
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0.3
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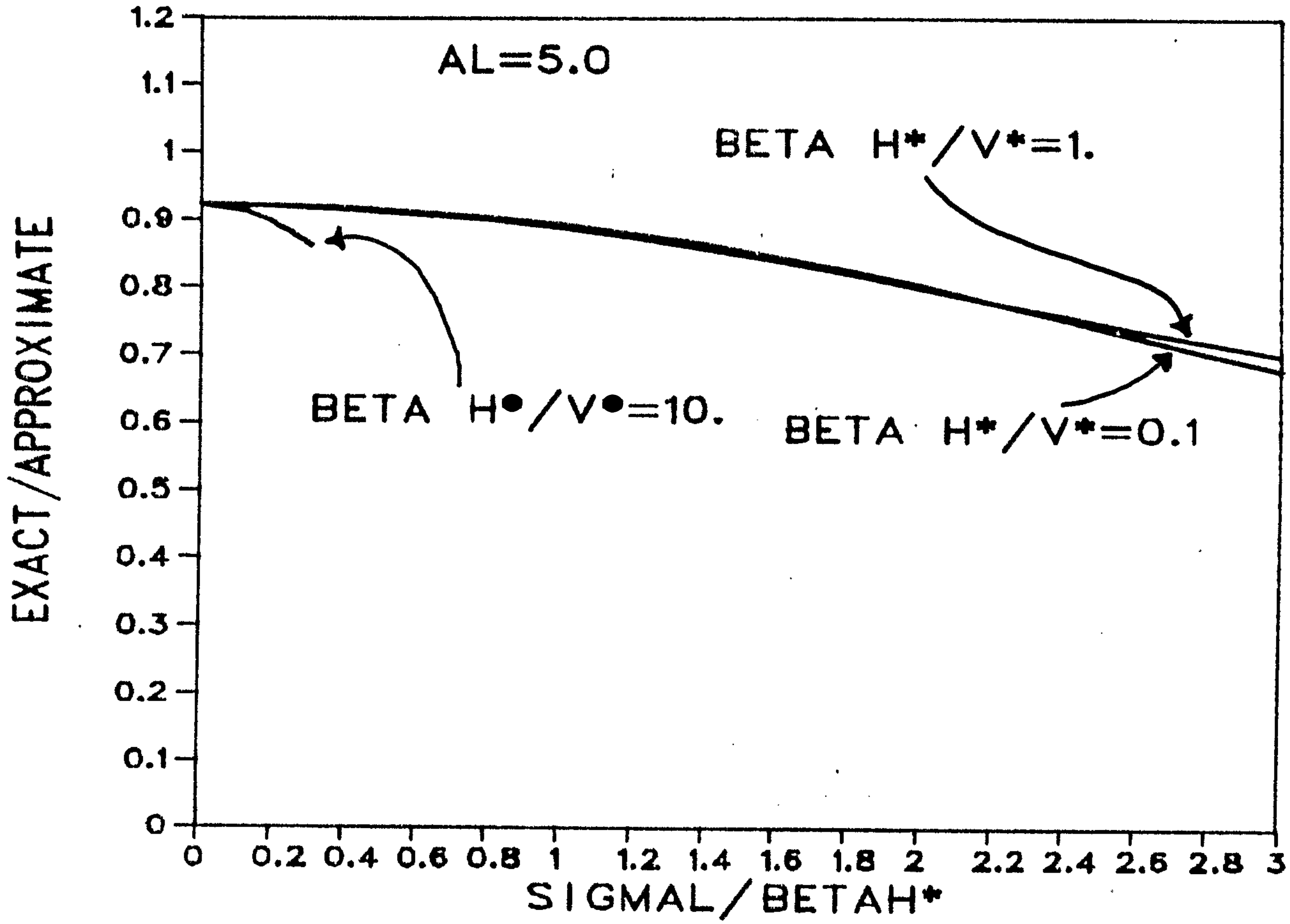
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2 2.2 2.4 2.6 2.8 3

Fig. 3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0.9
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Fig. 4